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1994 J. Phys. A: Math. Gen. 27 L125

(http://iopscience.iop.org/0305-4470/27/5/004)

J. Phys. A: Math. Gen. 27 (1994) L125–L129. Printed in the UK

LETTER TO THE EDITOR

On conservation laws and zero-curvature representations of the Liouville equation

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Received 23 December 1993

Abstract. Applying the first Noether theorem to the Liouville equation $u_{xy} = \exp u$, we find all (namely, a continuum of) non-trivial conservation laws of this equation. Then we find five new zero-curvature representations of the Liouville equation (by 2×2 traceless matrices) which contain, respectively, 1, 1, 2, 2 and 3 essential parameters. Finally, we show that all known zero-curvature representations of the Liouville equation are equivalent (in a definite sense) to matrices of conservation laws.

Recently, Wu *et al* [1] indicated a new interesting property of the Liouville equation

$$u_{xy} = \exp u. \tag{1}$$

They considered two *different* zero-curvature representations (by 2×2 and 3×3 traceless matrices) of (1) and derived thus two *different* infinite sequences of conservation laws for the Liouville equation. Conserved quantities were in involution within each of the two sequences, whereas the involution between the sequences was absent. Taking into account the result of [1] as well as that the two-dimensional Liouville field theory arises naturally at quantization of bosonic strings in all dimensions except 26 [2] and that a quantum treatment of the Liouville equation employs its Lax pair and conserved quantities [3], we see reasons for paying more attention to conservation laws and zero-curvature representations of (1). In this letter, we will find *all* conservation laws of the Liouville equation. We will see that (1) has a *continuum* of non-trivial conserved densities. Then we will find five new zero-curvature representations of the Liouville equation (by 2×2 traceless matrices) which contain, respectively, 1, 1, 2, 2 and 3 *essential* parameters. Finally, we will indicate gauge transformations which *trivialize* (in a definite sense) all known zero-curvature representations of (1). These results, however, should not evoke surprise, because the Liouville equation is very unusual in itself and differs from other integrable nonlinear equations in many respects. Indeed, (1) is known to possess the explicit general solution $u = \ln[2\phi_x\psi_y(\phi+\psi)^{-2}]$ with arbitrary $\phi(x)$ and $\psi(y)$, a non-commutative algebra of generalized symmetries with uncountable (continual) basis [4], and 4 continual classes of Bäcklund autotransformations [5].

Since the Liouville equation (1) is a *normal* Euler–Langrange system [6] with Lagrangian $L = \frac{1}{2}u_x u_y + \exp u$, we can take the opportunity of finding *all* its conservation laws by means of the first Noether theorem. We will use the Noether theorem in Olver's interpretation [6]. All generalized symmetries of (1) are known (in the form of their evolutionary representatives) [4, 7]:

$$u_\varepsilon = (D_x + u_x)a[x, v] + (D_y + u_y)b[y, w] \tag{2}$$

where $u_\varepsilon = \partial u/\partial\varepsilon$, ε is a symmetry group parameter, D_x and D_y are the total derivatives with respect to x and y, $a[x, v] = a(x, v, v_1, \ldots, v_m)$, $b[y, w] = b(y, w, w_1, \ldots, w_n)$, $v = u_{xx} - \frac{1}{2}u_x^2$, $v_k = D_x^k v$ $(k = 1, 2, \ldots)$, $w = u_{yy} - \frac{1}{2}u_y^2$, $w_k = D_y^k w$ $(k = 1, 2, \ldots)$, functions a and b and orders m and n are *arbitrary*. For employing the Noether theorem, we must find which of generalized symmetries (2) are variational symmetries [6]. Thus, all a and b must be determined such that $L_\varepsilon = \partial L/\partial\varepsilon$, calculated in accordance with (2), is a total divergence in x, y, u and derivatives of u.

Theorem 1. All variational symmetries of Lagrangian $L = \frac{1}{2}u_x u_y + \exp u$ are

$$u_\varepsilon = (D_x + u_x)\mathbf{E}_v(p[x, v]) + (D_y + u_y)\mathbf{E}_w(q[y, w]) \tag{3}$$

where functions $p[x, v] = p(x, v, v_1, \ldots, v_i)$ and $q[y, w] = q(y, w, w_1, \ldots, w_j)$ and orders i and j are *arbitrary*, $\mathbf{E}$ is the Euler operator, and index v or w shows that $\mathbf{E}$ is taken with respect to v or w (but not to u!).

Sketch of Proof. L_ε is a total divergence if, and only if, $\mathbf{E}_u(L_\varepsilon) = 0$ (for any function u). This is equivalent to

$$\mathbf{E}_u(v_y a[x, v] + w_x v[y, w]) = 0 \tag{4}$$

where $v_y = u_{xxy} - u_x u_{xy}$ and $w_x = u_{xyy} - u_y u_{xy}$. Differentiate (4) with respect to $\partial^{m+5}u/\partial x^{m+4}\partial y$ and get $[(-1)^{m+2} - 1]\partial a/\partial v_m = 0$, i.e. m is even, $m = 2i$, $i = 0, 1, 2, \ldots$. Differentiate (4) by $\partial^{2m+4}u/\partial x^{2m+4}$ and get $\partial^2 a/\partial v_m^2 = 0$ if $m > 0$. Differentiate (4) by $\partial^{2m+2}u/\partial x^{2m+2}$ and get $\partial^2 a/\partial v_m \partial v_{m-1} = 0$. And so on, up to $\partial^2 a/\partial v_m \partial v_{i+1} = 0$, $i = \frac{1}{2}m$. Take evident identity $\mathbf{E}_u(v_y \mathbf{E}_v(p[x, v])) = 0$ valid for arbitrary function $p(x, v, v_1, \ldots, v_i)$ and order i, subtract this identity from (4), choose $i = \frac{1}{2}m$ and $\partial^2 p/\partial v_i^2 = (-1)^i \partial a/\partial v_m$, and find that (4) is satisfied with functions $\tilde{a}[x, v]$ and $b[y, w]$, $\tilde{a} = a - \mathbf{E}_v(p)$, $\partial\tilde{a}/\partial v_m = 0$. Induction by m down to $m = 0$ proves $a = \mathbf{E}_v(p[x, v])$. Analogously, $b = \mathbf{E}_w(q[y, w])$. □

Before proceeding to conservation laws, let us consider variational symmetries (3) from the standpoint of the following *apparent* contradiction. On the other hand, (1) is evidently a normal system and must fall under the first Noether theorem therefore [6]. On the other hand, variational symmetries (3) are numbered by two arbitrary functions, therefore (1) should fall under the second Noether theorem which describes under-determined systems [6]. Which of the two statements is correct? Undoubtedly, the first one. Close examination of proof [6] of the second Noether theorem shows that applicability of the theorem demands dependence of variational symmetries on an arbitrary function of *all* independent variables. Since x and y are separated in (3), the arbitrariness required for variational symmetries of under-determined systems is not achieved, and the Liouville equation 'remains' a normal system. Therefore non-trivial variational symmetries (3) generate non-trivial conservation laws.

Theorem 2. Up to the equivalence [6], all conservation laws of (1) are

$$D_y p(x, v, v_1, \ldots, v_i) + D_x q(y, w, w_1, \ldots, w_j) = 0 \tag{5}$$

where functions p and q and orders i and j are *arbitrary*.

Sketch of proof. Take conservation laws in characteristic form $f\mathbf{E}_u(L) = 0$, replace characteristic f by the right-hand side of (3) due to the first Noether theorem, and represent the result as $D_y p + D_x q + D_y g + D_x h = 0$, where g and h are trivial, $g = 0$ and $h = 0$ for all u satisfying (1). □

It is *not* surprising that (1) has conservation laws (5), because $v_y = 0$ and $w_x = 0$ for all solutions u of (1). However, theorem 2 tells us more, namely, that *every* conservation law of the Liouville equation can be brought into form (5) by adding trivial conservation laws of the first and second kinds [6] to it. For example, $D_y(u_x^2) + D_x(-2\exp u) = 0$ does not belong to (5) by itself, but it is equivalent to $D_y(-2u_{xx} + u_x^2) + D_x 0 = 0$.

Proceeding to zero-curvature representations, let us recall the definition and main properties of them [8]. Consider the over-determined system of two linear equations $\Phi_x + A\Phi = 0$ and $\Phi_y + B\Phi = 0$, where $\Phi(x, y)$ is a k-component column, A and B are $k \times k$ matrix functions of independent variables x and y, dependent variables u and derivatives of u. This linear system is compatible if and only if

$$D_y A - D_x B - [A, B] = 0 \tag{6}$$

where square brackets denote the commutator. Compatibility condition (6) is said to represent a nonlinear system in u if all solutions u of the system satisfy (6). If one interprets matrices A and B as components of a connection and defines covariant derivatives of vectors Φ as $\nabla_x \Phi = (D_x + A)\Phi$ and $\nabla_y \Phi = (D_y + B)\Phi$, then (6) is nothing but condition $R = 0$ for curvature R defined as $R = [\nabla_y, \nabla_x] = D_y A - D_x B - [A, B]$. Localized transformations of vectors $\Phi' = S\Phi$ (S are $k \times k$ matrix functions of x, y, u and derivatives of u, $\det S \neq 0$) generate gauge transformations of matrices A and B

$$A' = SAS^{-1} - (D_x S)S^{-1} \qquad B' = SBS^{-1} - (D_y S)S^{-1} \tag{7}$$

and tensor transformations of the left-hand side of (6) $R' = SRS^{-1}$. Two zero-curvature representations, related by (7), should be considered as equivalent. One more equivalence is $A' = -A^T$, $B' = -B^T$ and $R' = -R^T$, where T denotes transposing. Only non-commuting traceless A and B should be considered, because (6) is a matrix of conservation laws when $[A, B] = 0$, and (6) splits into conservation law $D_y(\operatorname{tr} A) - D_x(\operatorname{tr} B) = 0$ plus (6) with new traceless $\tilde{A} = A - k^{-1} \operatorname{tr} A$ and $\tilde{B} = B - k^{-1} \operatorname{tr} B$ (tr denotes the trace). It is generally believed that only integrable systems admit non-commutative representations (6) where A and B contain an essential ('spectral') parameter which cannot be removed ('gauged out') by (7).

Though the problem of finding all zero-curvature representations of the Liouville equation is very attractive, we were unable to solve it in general form. However, the following special solution of the problem adds much to known unusual properties of (1).

Theorem 3. Up to equivalence $A' = SAS^{-1}$ and $B' = SBS^{-1}$ with any constant matrix S, the following 5 pairs (A, B) exhaust all zero-curvature representations (6) of (1) by 2×2 traceless non-commuting matrices $A(u_x)$ and $B(u)$:

$$A = \begin{pmatrix} \frac{1}{2}u_x & 1 \\ 0 & -\frac{1}{2}u_x \end{pmatrix} \qquad B = \begin{pmatrix} 0 & \alpha\exp(-u) \\ \frac{1}{2}\exp u & 0 \end{pmatrix} \tag{8}$$

$$A = \begin{pmatrix} \alpha & 0 \\ \frac{1}{2}u_x^2 - 2\alpha u_x & -\alpha \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ \exp u & 0 \end{pmatrix} \tag{9}$$

$$A = \begin{pmatrix} \alpha u_x & \alpha \\ (\frac{1}{2}-\alpha)u_x^2+\beta & -\alpha u_x \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ \exp u & 0 \end{pmatrix} \tag{10}$$

$$A = \begin{pmatrix} \frac{1}{2}\alpha u_x^2+\beta & 0 \\ \exp(2\alpha u_x) & -\frac{1}{2}\alpha u_x^2-\beta \end{pmatrix} \qquad B = \begin{pmatrix} \alpha \exp u & 0 \\ 0 & -\alpha \exp u \end{pmatrix} \tag{11}$$

$$A = \begin{pmatrix} \frac{1}{2}\alpha u_x^2+\beta & \exp(-2\alpha u_x) \\ \gamma \exp(2\alpha u_x) & -\frac{1}{2}\alpha u_x^2-\beta \end{pmatrix} \qquad B = \begin{pmatrix} \alpha \exp u & 0 \\ 0 & -\alpha \exp u \end{pmatrix} \tag{12}$$

where α, β and γ are arbitrary (complex) constants, $\alpha \neq 0$.

Sketch of proof. Take $A = A(u_x)$ and $B = B(u)$ in (6), replace u_{xy} by $\exp u$, and get equality $u_x^{-1}\partial A/\partial u_x - \exp(-u)\partial B/\partial u - [u_x^{-1}A, \exp(-u)B] = 0$ which must be an identity in u since u is any solution of (1). Take $\partial^2/\partial u \partial u_x$ of the identity, get $[M, N] = 0$, where $M = \partial(u_x^{-1}A)/\partial u_x$ and $N = \partial[\exp(-u)B]/\partial u$, and consider 3 possibilities: (i) $M = 0$, (ii) $N = 0$, and (iii) $M = r(u_x)C$, $N = s(u)C$, matrix C is constant, $C \neq 0$. □

This abundance of zero-curvature representations and free parameters cannot be reduced by gauge transformations (7).

Theorem 4. No two of zero-curvature representations listed in (8)–(12), neither coming from one of classes (8)–(12) at two different choices of parameters, nor coming from two different classes of (8)–(12) at any choice of parameters in each case, are related by gauge transformations (7).

Sketch of proof. Use quantities which are invariant under (7). Calculate R, $\nabla_x R$ and $\nabla_y R$ for (8)–(12), where $\nabla_x R = D_x R + [A, R]$ and $\nabla_y R = D_y R + [B, R]$. Invariants $\det R$, $\det(\nabla_x R)$ and $\det(\nabla_y R)$ prove the theorem except that α is essential in (9) and β is essential in (11). Then find that $\nabla_x R = cR$ for (9) and $[R, \nabla_x R] = d[R, \nabla_y R]$ for (11), where invariant c contains α and invariant d contains β. □

Nevertheless, gauge transformations (7) can reveal the origin of too numerous zero-curvature representations (8)–(12) of the Liouville equation. Each of pairs (A, B) (8)–(12) can be transformed by (7) into a pair (A', B') such that either A' or B' is *zero for all solutions* u of (1). Indeed, let us look at the following list of transforming matrices S and resultant matrices A' and B', the correspondence being $(n) \to (n+5)$, $n = 8, 9, \ldots, 12$:

$$S = \begin{pmatrix} 0 & \exp(-\frac{1}{2}u) \\ \exp(\frac{1}{2}u) & u_y \exp(-\frac{1}{2}u) \end{pmatrix} \qquad A' = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \qquad B' = \begin{pmatrix} 0 & \frac{1}{2} \\ -w+\alpha & 0 \end{pmatrix} \tag{13}$$

$$S = \begin{pmatrix} 1 & 0 \\ u_x & 1 \end{pmatrix} \qquad A' = \begin{pmatrix} \alpha & 0 \\ -v & -\alpha \end{pmatrix} \qquad B' = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \tag{14}$$

$$S = \begin{pmatrix} 1 & 0 \\ u_x & 1 \end{pmatrix} \qquad A' = \begin{pmatrix} 0 & \alpha \\ -v+\beta & 0 \end{pmatrix} \qquad B' = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \tag{15}$$

$$S = \begin{pmatrix} \exp(\alpha u_x) & 0 \\ 0 & \exp(-\alpha u_x) \end{pmatrix} \qquad A' = \begin{pmatrix} -\alpha v+\beta & 0 \\ 1 & \alpha v-\beta \end{pmatrix}$$

$$B' = \begin{pmatrix} \alpha z & 0 \\ 0 & -\alpha z \end{pmatrix} \tag{16}$$

$$S = \begin{pmatrix} \exp(\alpha u_x) & 0 \\ 0 & \exp(-\alpha u_x) \end{pmatrix} \qquad A' = \begin{pmatrix} -\alpha v + \beta & 1 \\ \gamma & \alpha v - \beta \end{pmatrix}$$
$$B' = \begin{pmatrix} \alpha z & 0 \\ 0 & -\alpha z \end{pmatrix} \tag{17}$$

where $z = \exp u - u_{xy}$, and v and w are the same as in (2). We see that $A' = 0$ in (13) and $B' = 0$ in (14)–(17) for all u satisfying (1). It is evident that zero-curvature representation (6) remains valid when we replace pair (A, B) by pair $(A + A_0, B + B_0)$ with any matrices A_0 and B_0 such that $A_0 = 0$ and $B_0 = 0$ for all solutions u. This kind of equivalence between zero-curvature representations resembles the procedure of adding trivial conservation laws of the first kind [6], whereas gauge transformations (7) resemble adding trivial conservation laws of the second kind (null divergences) [6]. Thus, all zero-curvature representations from theorem 3 are equivalent to commutative ones $D_x B' = 0$ or $D_y A' = 0$ which are matrices of conservation laws of (1), and the abundance of (8)–(12) should not therefore surprise. The well known 2×2 zero-curvature representation of (1) with matrix A of AKNS-type [1,9] belongs to our class (10) at $\alpha = \frac{1}{2}$ (up to equivalence (7) with constant S). As for the following 3×3 pair [1]

$$A = \begin{pmatrix} -\frac{1}{2}u_z & \alpha & 0 \\ 0 & 0 & \alpha \\ \alpha & 0 & \frac{1}{2}u_x \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & \frac{1}{2}\alpha^{-1}\exp u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{18}$$

we see the same picture as above:

$$S = \begin{pmatrix} 1 & 0 & \frac{1}{2}\alpha^{-1}u_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A' = \begin{pmatrix} 0 & \alpha & -\frac{1}{2}\alpha^{-1}v \\ 0 & 0 & \alpha \\ \alpha & 0 & 0 \end{pmatrix} \qquad B' = \begin{pmatrix} 0 & 0 & \frac{1}{2}\alpha^{-1}z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{19}$$

so that nothing contradicts the following conjecture.

Conjecture. Every zero-curvature representation (6) of the Liouville equation (1) is equivalent to commutative one $D_y P - D_x Q = 0$, $[P, Q] = 0$, where $P = P(x, v, v_1, \ldots, v_i)$ and $Q = Q(y, w, w_1, \ldots, w_j)$, the equivalence being $P = SAS^{-1} - (D_x S)S^{-1} + A_0$ and $Q = SBS^{-1} - (D_y S)S^{-1} + B_0$, where $A_0 = 0$ and $B_0 = 0$ for all solutions u of (1).

This work was supported by Grant Φ2-023 of the Fund for Fundamental Research, Republic of Belarus.

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